Cauchy's Root (Radical) Test

(B.Sc.-II, Paper-III)

Group- B

Sudip Kumar

Assistant Professor,

Department of Mathematics

Sachchidanand Sinha College

Aurangabad, Bihar

Cauchy's root (radical) test

Statement:→ If $\sum u_n$ be a series of positive term such that

$$\lim_{n\to\infty} (u_n)^{\frac{1}{n}} = l,$$ then the series

is (i) convergent if $l<1$

(ii) divergent if $l>1$

(iii) the test fails and the series may either converge or diverge if $l=1$.

Proof:→ Case (i) Let $l<1$.

$\because u_n > 0, \forall n \Rightarrow l \geq 0$

We can choose $\epsilon > 0$ such that

$$0 < l+\epsilon < 1$$

$$\because \lim_{n\to\infty} u_n^{\frac{1}{n}} = l$$

$\therefore \exists$ a natural number $N \in \mathbb{N}$ s.t.

$$l-\epsilon < u_n^{\frac{1}{n}} < l+\epsilon, \ \forall \ n \geq N.$$

$$\therefore u_n < (l+\epsilon)^n, \ \forall \ n \geq N$$

But $\sum_{n=N}^{\infty} (l+\epsilon)^n$ is a G.P. series with c.r $= l+\epsilon$.

Where, $0 < l+\epsilon < 1$.

$\therefore \sum_{N}^{\infty} (l+\epsilon)^n$ is convergent.

And so $\sum_{N}^{\infty} u_n$ is also convergent.

$\therefore$ By comparision test

$\sum_{1}^{\infty} u_n$ is convergent.

($\because$ addition or ommision of finite number of terms does not affect the convergence or divergence of the series)

Case(ii) Let $l > 1$,

Then we can choose $\epsilon > 0$ s.t.

$l - \epsilon > 1$.

As in the case (i) $\exists$ a natural number $N \in \mathbb{N}$ s.t.

$$u_n^{\frac{1}{n}} > l - \epsilon, \quad \forall\, n \geq N(\epsilon)$$

$$\therefore u_n > (l-\epsilon)^n$$

But $\sum_{N}^{\infty} (l-\epsilon)^n$ is a G.P. series with

c.r. $= (l-\epsilon) > 1$, and so it divergent

$\therefore \sum u_n$ is also divergent by comparision test.

<u>Case (iii)</u> Consider two series $\Sigma \frac{1}{n}$ and $\Sigma \frac{1}{n^2}$.

$\because$ The series $\Sigma \frac{1}{n}$ is divergent

while $\lim_{n\to\infty} \left(\frac{1}{n}\right)^{\frac{1}{n}} = 1$ ($\neq 0$ & finite)

And the series $\Sigma \frac{1}{n^2}$ converges.

while $\lim_{n\to\infty} \left(\frac{1}{n^2}\right)^{\frac{1}{n}} = 1$ ($\neq 0$ & finite)

Thus the test fails to give any definite information for $l = 1$.

#.

Cauchy's root test

Remark: → (i) $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} = e$

(ii) $\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{n} = \frac{1}{e}$

(iii) $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{-n} = \frac{1}{e}$

(iv) $\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{-n} = e$

Example ① Test for convergence the series Σu_n, whose nth term is

$$u_n = \frac{n^{n^2}}{(1+n)^{n^2}}.$$

Solution: →

Here, $u_n = \dfrac{n^{n^2}}{(1+n)^{n^2}}$

$$\therefore (u_n)^{\frac{1}{n}} = \left(\frac{n}{1+n}\right)^{n} = \frac{1}{\left(1+\frac{1}{n}\right)^{n}}$$

$$\therefore \lim_{n\to\infty}(u_n)^{\frac{1}{n}} = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^{n}} = \frac{1}{e} < 1.$$

∴ The series Σu_n is the given series is convergent.

Example②:→ Test the convergency of the series $\sum (1+\frac{1}{n})^{n^2}$.

Solution:→

Here $u_n = (1+\frac{1}{n})^{n^2}$

$$\therefore (u_n)^{\frac{1}{n}} = \left[(1+\frac{1}{n})^{n^2}\right]^{\frac{1}{n}} = (1+\frac{1}{n})^{n}.$$

$$\therefore \lim_{n\to\infty} u_n^{\frac{1}{n}} = \lim_{n\to\infty} (1+\frac{1}{n})^{n} = e > 1$$

∴ The given series $\sum u_n$ is divergent.

Example③:→ Test the convergency of the series $\sum_{n=1}^{\infty} (\sqrt[n]{n} - 1)^n$

Solution:→ Here $u_n = (n^{\frac{1}{n}} - 1)^n$, then

$$\lim_{n\to\infty} (u_n)^{\frac{1}{n}} = \lim_{n\to\infty} (n^{\frac{1}{n}} - 1)$$

$$= \lim_{n\to\infty} n^{\frac{1}{n}} - 1$$

$$= 1 - 1 = 0, (\neq 1) \text{ and less than } 1.$$

Hence by Cauchy's root test.

The given series is convergent.

Thank you